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LATERAL-TORSIONAL FLUTTER OF A DEEP CANTILEVER LOADED BY A LATERAL FOLLOWER FORCE AT THE TIP

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1. INTRODUCTION

As pointed out by Bolotin [1, 2], the study of the stability of structures under follower force systems apparently started with work by Nikolai in the late 1920s. Now there are many papers and even a few books devoted to this field; see, for example, the work of references [1]–[7]. Since the beginning of this period, much of the analytical research has focused on the effects of various physical phenomena, such as damping and transverse shear deformation, on the stability of beams subjected to various types of follower forces.

In spite of all the published work, there seems to be very little literature concerned with the lateral-torsional stability of deep cantilevered beams loaded by a transverse follower force at the tip. This problem has some practical applications, such as the effect of jet engine thrust on the aeroelastic flutter of a flexible wing. According to Bolotin [1], this type of system was first considered by himself in reference [8]. Although the analysis presented therein is applicable to the tip-loaded cantilever case, no results specific to that case were presented. Como [9] analyzed a cantilevered beam subjected to a lateral follower force at the tip. The distributed mass and inertia properties of the beam were neglected, although a concentrated mass and inertia at the tip were included. Without neglecting the distributed mass and inertia properties of the beam, Wohlhart [10] undertook an extensive study, and results for a wide variation of several parameters were presented in this truly excellent paper. The results of the present study agree with those of Wohlhart. Later work by Feldt and Herrmann [11] added the influence of fluid flowing past a wing, the structural properties of which are represented by a beam undergoing bending and torsion. Unfortunately, the results for cases in which the aerodynamic forces are neglected agree with neither those of reference [10] nor those of the present work. Other than these three papers, to the best of the author's knowledge, this problem appears to have received no further attention in the literature. It is the objective of the present paper to consider further this non-conservative elastic stability problem and present a few results and observations that go beyond those of reference [10].

We first develop a weak form of the partial differential equations of motion for a deep, symmetric beam under the action of a tip follower force acting in the plane of symmetry. Then an approximate solution using cantilever beam bending and torsional modes is obtained. The effects of three parameters are investigated: the ratio of the uncoupled fundamental bending and torsional frequencies and dimensionless parameters reflecting the mass radius of gyration and the offset from the elastic axis of the mass centroid.

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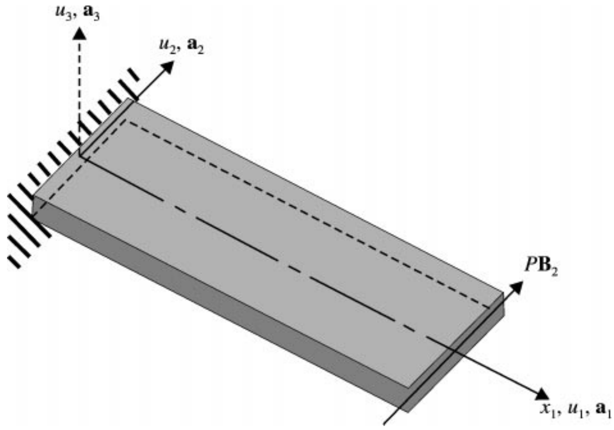

Figure 1. Schematic of wing showing co-ordinate systems and follower force.

2. EQUATIONS OF MOTION

Consider a cantilevered beam with torsional stiffness GJ and bending stiffnesses EI_2 and EI_3 with $EI_3 \gg EI_2$. It is noted here that the bending analysis neglects transverse shear deformation and rotary inertia, and the torsional analysis neglects the warping restraint; thus, the beam theory is strictly along classical lines. The Cartesian co-ordinates x_1, x_2, and x_3 are along the elastic axis and the two transverse directions, respectively, as shown in Figure 1. For the purpose of analysis, we introduce two sets of dextral triads of unit vectors. The unit vectors of the first set, $\mathbf{a}_i$ with $i = 1$, 2, and 3, are parallel to x_i and fixed in an inertial frame. Those of the other set are fixed in the local cross-sectional frame of the deformed beam and denoted by $\mathbf{B}_i(x_1, t)$, with $i = 1$, 2, and 3. Denote the displacements along $\mathbf{a}_i$ as $u_i(x_1, t)$ with $i = 1$, 2, and 3; and denote the section rotation due to torsion as $\theta_1(x_1, t)$. The load is directed along unit vector $\mathbf{B}_2(\ell, t)$, where $\mathbf{B}_2(\ell, t) = -u_2'(\ell, t)\mathbf{a}_1 + \mathbf{a}_2 + \theta_1(\ell, t)\,\mathbf{a}_3$; here $()'$ denotes a partial derivative with respect to x_1. Thus, the virtual work done by this force through a virtual displacement is

$$\begin{aligned} \overline{\delta W} &= P\mathbf{B}_2(\ell, t)\,[\delta u_1(\ell, t)\mathbf{a}_1 + \delta u_2\,(\ell, t)\mathbf{a}_2 + \delta u_3(\ell, t)\mathbf{a}_3] \\ &= P(-u_2'\delta u_1 + \delta u_2 + \theta_1\delta u_3)\,|_0^\ell\,. \end{aligned} \tag{1}$$

In keeping with the non-conservative nature of the follower force, there exists no potential energy which, upon variation, will yield this expression for the virtual work. We will subsequently ignore the longitudinal displacement u_1.

For a beam subject to a bending moment $\bar{M}_3$ that is constant in time but varying in x_1, and in which deflections due to that moment are ignored (since $EI_3 \gg EI_2$), the strain energy can be written as

$$U = \int_0^\ell \left[\frac{GJ}{2}\,\theta_1'^2 + \frac{EI_2}{2}\,u_3''^2 + \bar{M}_3\,(u_2'' + \theta_1\,u_3'')\right]\mathrm{d}x_1\,. \tag{2}$$

To find the equilibrium state of deformation in the beam, one may consider only the first order terms in $\delta U - \overline{\delta W}$, such that

$$\int_0^\ell (\bar{M}_3\delta u_2'' - P\delta u_2')\,\mathrm{d}x_1 = 0. \tag{3}$$

Thus,

$$\bar{M}_3 = P(\ell - x_1) \tag{4}$$

as expected. To obtain a weak form that governs the behavior of small static perturbations about the equilibrium state, one may set the second order terms in $\delta U - \overline{\delta W}$ equal to zero, so that

$$\delta U - \overline{\delta W} = \int_0^{\ell} [EI_2 u_3'' \delta u_3'' + GJ\theta_1' \delta\theta_1' + P(\ell - x_1)(\theta_1 \delta u_3'' + u_3'' \delta\theta_1)] \, \mathrm{d}x_1 - P\theta_1 \delta u_3|_0^{\ell} = 0. \tag{5}$$

Integration by parts can eliminate the trailing term, so that

$$\delta U - \overline{\delta W} = \int_0^{\ell} \{EI_2 u_3'' \delta u_3'' + GJ\theta_1' \delta\theta_1' + P(\ell - x_1) u_3'' \delta\theta_1 + P[(\ell - x_1)\theta_1]'' \delta u_3\} \, \mathrm{d}x_1 = 0. \tag{6}$$

It can be shown that there is no value of P that will result in buckling. To proceed with an investigation of the dynamic stability, one may now add the variation of the kinetic energy and, using Hamilton's principle, consider the stability of small vibrations about the static equilibrium state.

The kinetic energy of the vibrating beam is simply

$$K = \frac{1}{2} \int_0^{\ell} (m\dot{u}_2^2 + m\dot{u}_3^2 + m\bar{\sigma}^2 \dot{\theta}_1^2 + 2m\bar{e}\dot{\theta}_1 \dot{u}_3) \, \mathrm{d}x_1, \tag{7}$$

where $\dot{(\ )}$ is a partial derivative with respect to time, m the mass per unit length, $\bar{e}$ the offset in the $\mathbf{a}_2$ direction of the mass centroid from the reference line, and $\bar{\sigma}$ the cross-sectional mass radius of gyration. We now undertake a straightforward application of Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta U - \overline{\delta W} - \delta K) \, \mathrm{d}t = 0, \tag{8}$$

where t_1 and t_2 are fixed times. Integrating by parts in time, setting δu_3 and $\delta\theta_1$ equal to zero at the end of the time interval, removing the time integration, assuming that the motion variables are proportional to $\mathrm{e}^{\bar{s}t}$, and introducing a set of non-dimensional variables, such that

$$\begin{gathered} (\ )' = \frac{\mathrm{d}(\ )}{\mathrm{d}x}, \quad x_1 = x\ell, \qquad u_3 = \ell w \exp(\bar{s}t), \qquad \theta_1 = \theta \exp(\bar{s}t), \\ p = \frac{P\ell^2}{\sigma EI_2}, \quad s^2 = \frac{m\ell^4 \bar{s}^2}{EI_2}, \qquad \sigma = \frac{\bar{\sigma}}{\ell}, \quad e = \frac{\bar{e}}{\ell}, \qquad r^2 = \frac{EI_2 \beta_1^4 \sigma^2}{GJ\gamma_1^2}, \end{gathered} \tag{9}$$

one obtains a weak form governing the flutter problem

$$\int_0^1 \left\{ w''\delta w'' + \sigma p[(1-x)\theta]''\delta w + s^2(w+e\theta)\delta w + \frac{\beta_1^4}{r^2\gamma_1^2}\theta'\delta\theta' + \frac{p}{\sigma}(1-x)w''\delta\theta + s^2\left(\frac{e}{\sigma^2}w + \theta\right)\delta\theta \right\} \mathrm{d}x = 0, \tag{10}$$

where the dimensionless parameters e and σ govern the offset of the mass centroid and mass radius of gyration respectively. The dimensionless parameter r is the ratio of the fundamental bending and torsion frequencies of the unloaded beam with $e = 0$.

3. APPROXIMATE SOLUTION AND RESULTS

This weak form can be solved approximately by assuming a set of uncoupled cantilever beam free-vibration modes for bending and torsion. To obtain converged results for the range of parameters considered, four of each type were found to be adequate. Specifying values for r, e, and σ, one can solve for the real and imaginary parts of s as functions of p. Depending on the values chosen for these parameters, flutter will occur when the imaginary parts of two modes coalesce. The modes that coalesce can be traced back to modes that for $p = 0$ are either two bending modes, two torsion modes, or one of each.

It can easily be shown that when $e = 0$, the eigenvalues do not depend on σ. This surprising result makes it possible to characterize the critical load in terms of only one parameter, r. It should be noted, however, that the mode shapes are not independent of σ when $e = 0$. In a typical case, the real parts of all eigenvalues are zero for sub-critical values of p. The imaginary parts of the eigenvalues depend only on p and r when $e = 0$, and on p, r, σ and e when $e \neq 0$. At the point when coalescence occurs, the real part of one mode becomes negative, while the real part of another becomes positive.

Example results for $e = 0$ are shown in Figures 2–6. In Figure 2 the imaginary parts of the four smallest eigenvalues are shown versus p for a large value of $r = 3{\cdot}8$. In this case, at $p = 0$, two modes that start out as the first two torsional modes coalesce. Notice that the next higher modes, the first bending and third torsion modes at $p = 0$, coalesce for just a slightly larger value of p. If r is taken to be a little larger, a complicated pattern emerges

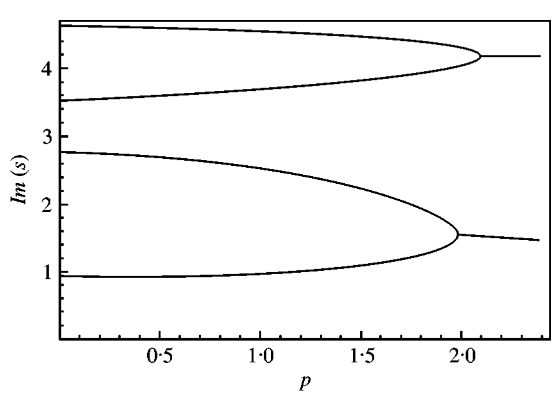

Figure 2. Imaginary parts of the four smallest eigenvalues versus p for $r = 3{\cdot}8$ and $e = 0$.

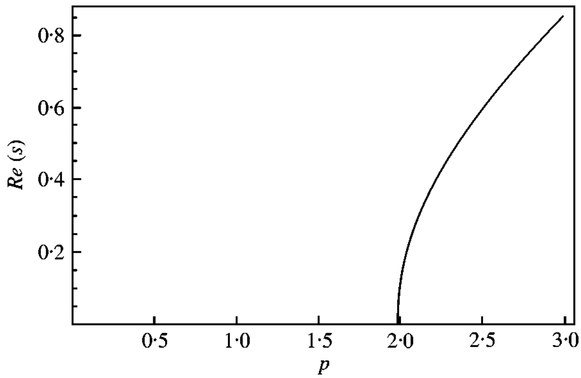

Figure 3. Real part of the eigenvalue for the unstable mode versus p for $r = 3{\cdot}8$ and $e = 0$.

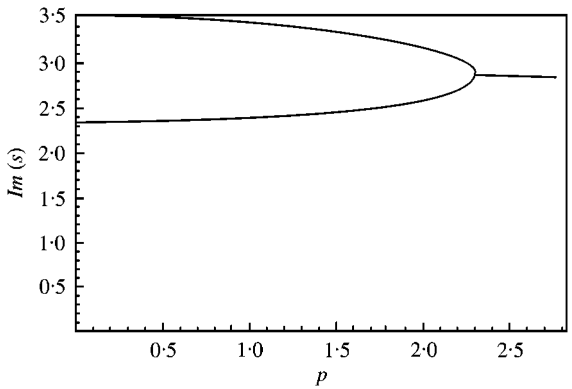

Figure 4. Imaginary parts of the two smallest eigenvalues versus p for $r = 3/2$ and $e = 0$.

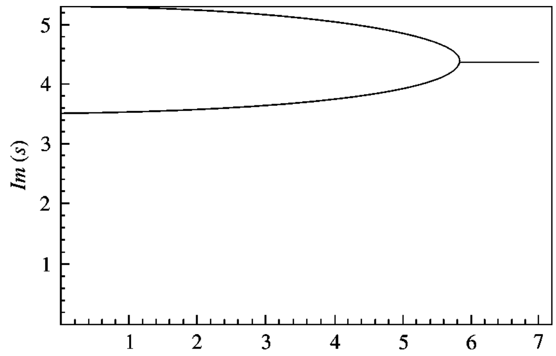

Figure 5. Imaginary parts of the two smallest eigenvalues versus p for $r = 2/3$ and $e = 0$.

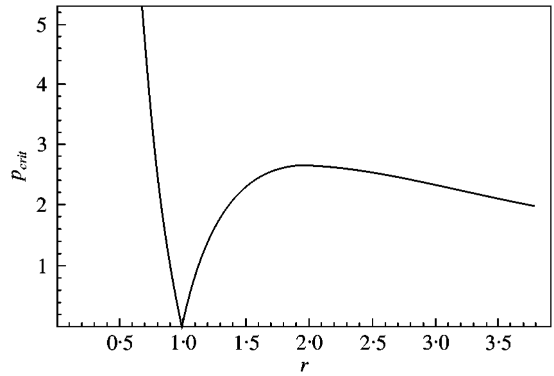

Figure 6. Critical load for $e = 0$ versus r.

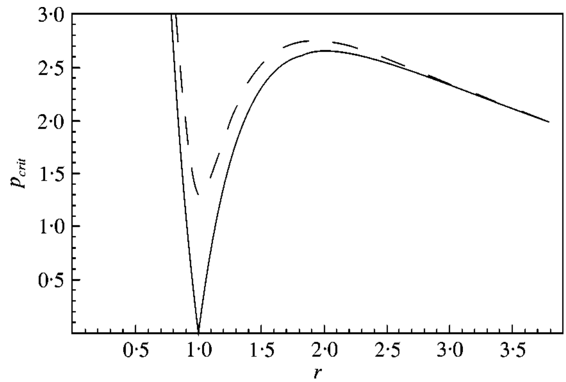

Figure 7. Critical load versus r for $\sigma = 0{\cdot}05$ with $e = 0$ (——) and 0·005 (– – –).

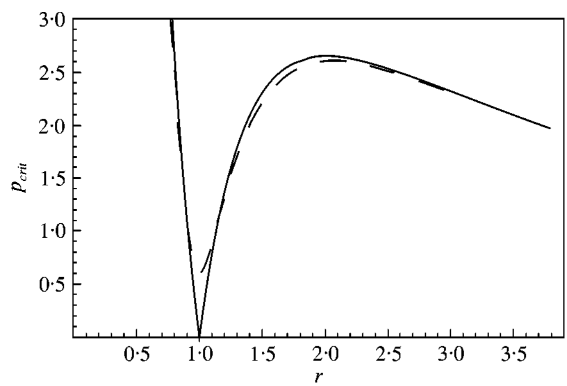

Figure 8. Critical load versus r for $\sigma = 0{\cdot}05$ with $e = 0$ (——) and $-0{\cdot}005$ (– – –).

with the critical load, because multiple torsional modes occur below the first bending mode, and the critical coalescence may jump up to the second two modes. Figure 3 shows the usual zero real part up to the coalescence and positive real part thereafter. More realistic values

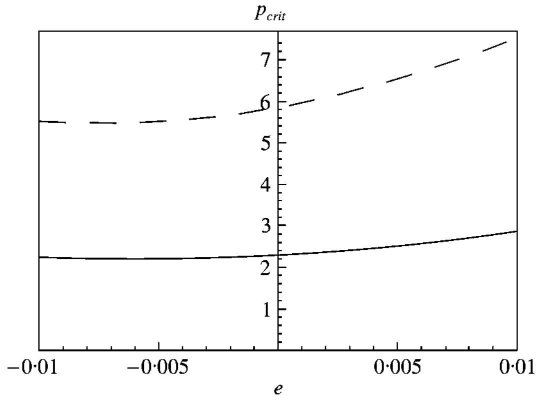

Figure 9. Critical load versus e for $\sigma = 0{\cdot}05$ with $r = 3/2$ (——) and 2/3 (– – –).

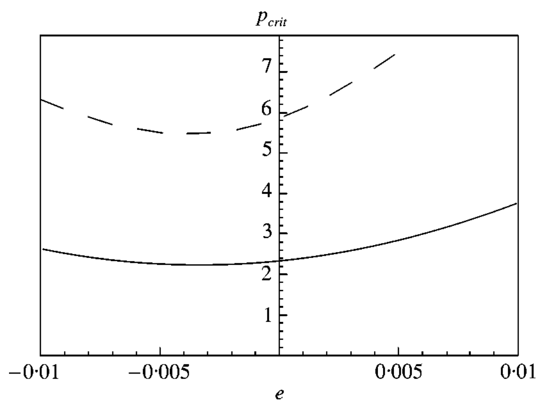

Figure 10. Critical load versus e for $\sigma = 0{\cdot}025$ with $r = 3/2$ (——) and 2/3 (– – –).

are shown in Figures 4 and 5. In the former case, the first bending mode is above the first bending, and in the latter it is below. As r becomes smaller still, the critical load greatly increases. The variation of the critical load versus r for $e = 0$ is shown in Figure 6. Notice that an arbitrarily small force destabilizes the system when $r = 1$; similar observations were made in reference [2, p. 349], and reference [10].

For a typical value of $\sigma = 0{\cdot}05$, small values of the mass centroid offset parameter e do not change the results qualitatively, except for removing the "cusp" in the plot of p_{crit} versus r at $r = 1$. Rather than a cusp at $r = 1$, when $e \neq 0$ one finds a smooth curve that has a non-zero minimum value. Plots for $e = 0{\cdot}005$ and $-0{\cdot}005$ are shown in Figures 7 and 8, respectively, each also showing results for $e = 0$. One finds a greater qualitative change for positive values of e than for negative values. The variation of the results versus e for typical values of $r = 2/3$ and 3/2 is shown in Figure 9 for $\sigma = 0{\cdot}05$ and in Figure 10 for $\sigma = 0{\cdot}025$. Note the increased sensitivity of the critical load versus e curve for the smaller value of σ and for positive values of e. Finally, we consider the variation of p_{crit} versus σ for typical values of $r = 2/3$ and 3/2. Figure 11 shows the variation of p_{crit} versus σ for $e = 0{\cdot}005$ and Figure 12 for $e = -\,0{\cdot}005$. One sees p_{crit} decreasing with increasing σ for both positive and negative

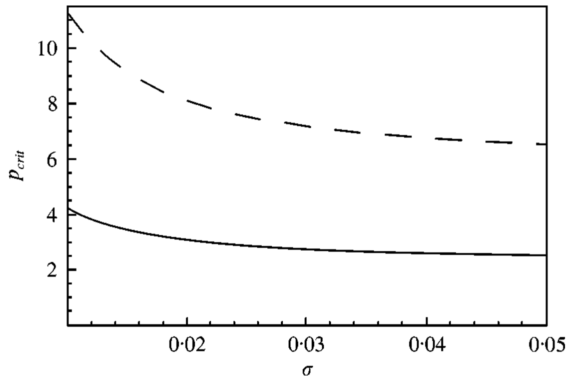

Figure 11. Critical load versus σ for $e = 0{\cdot}005$; $r = 3/2$ (——) and 2/3 (- - -).

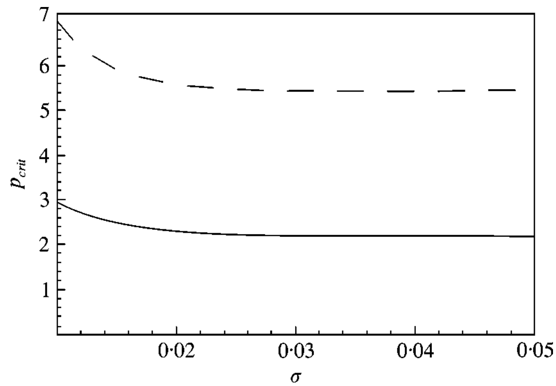

Figure 12. Critical load versus σ for $e = -0{\cdot}005$; $r = 3/2$ (——) and 2/3 (- - -).

values of e. For the values of r chosen for these plots, the curve p_{crit} versus σ becomes rather flat as σ becomes large. Depending on the value of r, these flat regions can tend monotonically and asymptotically to the $e = 0$ value of p_{crit} or they may reverse while converging asymptotically. The value of r governs the types of motion that make up the two lowest modes.

4. CONCLUSION

This note presents a numerical study of the flutter instability associated with a lateral follower force acting on the tip of a deep cantilever beam. This problem may have application to high-aspect-ratio wings loaded by jet engines.

The present study has focused on a uniform beam without bending-torsion coupling. There are three non-dimensional parameters that govern the dimensionless critical load: e (the ratio of the cross-sectional mass center offset from the elastic axis to the beam length), σ (the ratio of the cross-sectional mass radius of gyration to the beam length), and r (the ratio of the fundamental bending and torsional frequencies of an unloaded and uncoupled beam). Remarkably, when $e = 0$ the problem ceases to be dependent on σ, and the critical load depends only on r. When $e \neq 0$, there is a rich dependency of the critical load on both e and σ.

One could generalize this work to include bending-torsion elastic coupling of the beam, tip mass/inertia, and aeroelastic effects.

REFERENCES

1. V. V. BOLOTIN 1963 *Nonconservative Problems of the Theory of Elastic Stability*. New York: Pergamon Press.
2. V. V. BOLOTIN 1964 *Dynamic Stability of Elastic Systems*. San Francisco: Holden-Day, Inc.
3. H. H. E. LEIPHOLZ 1978 in *Variational Methods in the Mechanics of Solids* (S. Nemat-Nasser, editor) 151–155. Oxford: Pergamon Press. On variational principles for non-conservative mechanical systems with follower forces.
4. Z. CELEP 1979 *Journal of Sound and Vibration* **64**, 173–178. On the lateral stability of a cantilever beam subjected to a non-conservative load.
5. Y. P. PARK 1987 *Journal of Sound and Vibration* **113**, 407–415. Dynamic stability of a free Timoshenko beam under a controlled follower force.
6. L.-W. CHEN and D.-M. KU 1992 *Journal of Sound and Vibration* **153**, 403–411. Eigenvalue sensitivity in the stability analysis of beck's column with a concentrated mass at the free end.
7. K. HIGUCHI 1994 *Proceedings of the* 35*th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Hilton Head, South Carolina*, 2402–2408. An experimental model of a flexible free-free column in dynamic instability due to an end thrust.
8. V. V. BOLOTIN 1959 *Kolebaniia v turbomashinakh*, pp. 23–42. U.S.S.R.: Izdatel'stvo Akademii Nauk. On vibrations and stability of bars under the action of non-conservative forces.
9. M. COMO 1966 *International Journal of Solids and Structures* **2**, 515–523. Lateral buckling of a cantilever subjected to a transverse follower force.
10. K. WOHLHART 1971 *Zeitschrift fuer Flugwissenschaften* **19**, 291–298. Dynamische Kippstabilität eines Platenstreifens unter Folgelast.
11. W. T. FELDT and G. HERRMANN 1974 *Journal of the Franklin Institute* **297**, 467–478. Bending-torsional flutter of a cantilevered wing containing a tip mass and subjected to a transverse follower force.